Distributions Theory

Binomial Distributions

This distribution was discovered by James Bernoulli in 1700. Consider a random experiment with two possible outcomes which we call success and failure. Let p be the probability of success and $q = 1 - p$ be the probability of failure. p is assumed to be fixed from trial to trial. Let X denote the number of success in n independent trials. Then X is a random variable and may take the values $0, 1, 2, ..., n$.

Then, $P(X = x) = P(x$ successes and $n - x$ failures in n repetitions of the experiment$)$

There are ${}^nC_x$ mutually exclusive ways each with probability $p^x\, q^{n-x}$, for the happening of x successes out of n repetitions of the experiment. Hence

$$P(X = x) = {}^n C_x\, p^x\, q^{n-x}\,, \quad x = 0, 1, 2, ...$$

Definition

A random variable X is said to follow the Binomial distribution with parameters n and p if the pmf is,

$$P(X = x) = \begin{cases} {}^nC_x\, p^x\, q^{n-x} & x = 0, 1, 2, ...; 0 < p < 1; p + q = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Mean: $E(X) = \sum x\, P(X = x) = \sum_{x=0}^{n} x \quad {}^nC_x\, p^x\, q^{n-x}$

$$\Rightarrow p \sum_{x=1}^{n} x\, \frac{n}{x} \quad {}^{n-1}C_{x-1}\, p^{x-1}\, q^{n-x} = n\, p \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1}\, q^{(n-1)-(x-1)}$$

$$\Rightarrow n\, p\, (p + q)^{n-1} = n\, p\, (1)^{n-1} = n\, p$$

So, **Mean** $= E(X) = \mu_1' = n\, p$

$$E(X^2) = \sum x^2\, P(X = x) = \sum_{x=0}^{n} x^2 \;\; {}^nC_x\, p^x\, q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{n} (x\,(x-1) + x) \;\; {}^nC_x \;\; p^x\, q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{n} x\,(x-1) \;\; {}^nC_x \;\; p^x\, q^{n-x} + \sum_{x=0}^{n} x \;\; {}^nC_x\, p^x\, q^{n-x}$$

$$\Rightarrow n\,p + p^2 \sum_{x=2}^{n} x\,(x-1)\ \frac{n(n-1)}{x(x-1)}\ \ {}^{n-2}C_{x-2}\ \ p^{x-2}\,q^{n-x}$$

$$\Rightarrow n\,p + n(n-1)\,p^2 \sum_{x=2}^{n}\ {}^{n-2}C_{x-2}\ \ p^{x-2}\,q^{(n-2)-(x-2)}$$

$$\Rightarrow n\,p + n(n-1)\,p^2\,(p+q)^{n-2} = n\,p + n(n-1)\,p^2\,(1)^{n-2}$$

$$\Rightarrow n\,p + n(n-1)\,p^2$$

So, $E(X^2) = \mu_2' = n\,p + n(n-1)\,p^2$

Variance$= \mu_2 = \mu_2' - \mu_1'^2 = n\,p + n(n-1)\,p^2 - (n\,p)^2 = n\,p\,(1-p)$

Similarly, $E(X^3) = \mu_3' = \sum x^3\,P(X=x)$

$$\Rightarrow \sum \Big(x\,(x-1)\,(x-2) + 3\,x^2 - 2\,x\Big)$$

$$\Rightarrow n\,(n-1)\,(n-2)\,p^3 + 3\,n\,(n-1)\,p^2 + n\,p$$

Similarly, $E(X^4) = \mu_4' = \sum x^4\,P(X=x)$

$$\Rightarrow n\,(n-1)\,(n-2)\,(n-3)\,p^4 + 3\,n\,(n-1)\,(n-2)\,p^3 + 7\,n\,(n-1)\,p^2 + n\,p$$

Moment Generating Function

The m.g.f, $M_X(t) = E(e^{t\,X}) = \sum e^{t\,x}\,P(X=x) = \sum_{x=0}^{n} e^{t\,x}\ \ {}^{n}C_x\,p^x\,q^{n-x}$

$$\Rightarrow \sum_{x=0}^{n}\ {}^{n}C_x\,(p\,e^t)^x\,q^{n-x} =\ (q+p\,e^t)^n$$

$$E(X) = \mu_1' = \left.\frac{d\,M_X(t)}{dt}\right|_{t=0} = \left.n\,(q+p\,e^t)^{n-1}\,p\,e^t\,\right|_{t=0}$$

$$\Rightarrow n\,p\,(q+p)^{n-1} = n\,p$$

$$E(X^2) = \mu_2' = \left.\frac{d^2\,M_X(t)}{dt^2}\right|_{t=0} = \left.\frac{d}{dt}\left\{\frac{d\,M_X(t)}{dt}\right\}\right|_{t=0}$$

$$\Rightarrow \left.\frac{d}{dt}\left\{n\,(q+p\,e^t)^{n-1}\,p\,e^t\,\right\}\right|_{t=0}$$

$$\Rightarrow n\,(n-1)\,(q+p\,e^t)^{n-2}\,(p\,e^t)^2 + n\,(q+p\,e^t)^{n-1}\,p\,\Big|_{t=0}$$

$$\Rightarrow n\,(n-1)\,(q+p)^{n-2}\,p^2 + n\,(q+p)^{n-1}\,p\; = n\,(n-1)\,p^2 + n\,p$$

Additive property of the binomial distribution

If X is a $B(n_1,\, p)$ and Y is a $B(n_2,\, p)$ and they are independent then their sum $X+Y$ also follows $B(n_1+n_2,\, p)$.

Proof: Since $X \sim B(n_1,\, p)$, $M_X(t) = (q+p\,e^t)^{n_1}$

$Y \sim B(n_2,\, p)$, $M_Y(t) = (q+p\,e^t)^{n_2}$

$M_{(X+Y)}(t) = M_X(t)\,M_Y(t)$, Since X and Y are independent.

$$\Rightarrow M_{(X+Y)}(t) = (q+p\,e^t)^{n_1}\,(q+p\,e^t)^{n_2} = (q+p\,e^t)^{n_1+n_2}$$

$\Rightarrow$ M.G.F. of $B(n_1+n_2,\, p)$, So, $X+Y \sim B(n_1+n_2,\, p)$

Ex.: The mean and variance of a binomial random variable X are 12 and 6 respectively. **(i)** Find $P(X=0)$ **(ii)** Find $P(X>1)$

Sol: Let $X \sim B(n,p)$, Given $E(X) = n\,p = 12$ and $V(X) = n\,p\,q = 6$.

$$\frac{npq}{np} = \frac{6}{12} \Rightarrow q = \frac{1}{2} \text{ so, } p = 1-q = 1-\tfrac{1}{2} = \tfrac{1}{2}$$

and $n\,\frac{1}{2} = 12 \;\Rightarrow\; n = 24$

(i) $P(X=0) = {}^{24}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{24-0} = \left(\frac{1}{2}\right)^{24}$

(ii) $P(X>1) = 1 - P(X \le 1) = 1 - \sum_{x=0}^{1} {}^{24}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{24-x} = 1 - \sum_{x=0}^{1} {}^{24}C_x \left(\frac{1}{2}\right)^{24} = 1 - \left(\frac{1}{2}\right)^{24}\left({}^{24}C_0 + {}^{24}C_1\right) = 1 - 25\left(\frac{1}{2}\right)^{24}$

Poisson Distributions

Poisson distribution is a discrete probability distribution. This distribution was developed by the French mathematician Simeon Denis Poisson in 1837. This distribution is used to represent rare events. Poisson Distribution is a limiting case of binomial distribution under certain conditions.

Definition

A discrete random variable X is said to follow the Poisson distribution with parameters λ if the pmf is,

$$P(X = x) = \begin{cases} \dfrac{\lambda^x \, e^{-\lambda}}{x!} & x = 0, 1, 2, ...; \lambda > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Poisson distribution as a limiting case of Binomial:

The Poisson distribution is obtained as an approximation to the binomial distribution under the conditions

(i) n is very large, **(ii)** p is very small, **(iii)** $n\,p = \lambda$ a finite quantity.

Proof. Let $X \sim B(n,\, p)$ then $P(X = x) = {}^nC_x \, p^x \, q^{n-x}$, $x = 0, 1, 2, ...; 0 < p < 1; p + q = 1;$

$$\Rightarrow \frac{n!}{x! \, (n - x!)} \, p^x \, q^{n-x} = \frac{n(n-1)...(n-(x-1))}{x!} \, p^x \, q^{n-x}$$

$$\Rightarrow \frac{n^x \left(1 - \frac{1}{n}\right) ... \left(1 - \frac{(x-1)}{n}\right)}{x!} \, p^x \, (1-p)^{n-x}$$

$$\text{Now, } \lim_{n \to \infty} \left(1 - \frac{1}{n}\right) ... \left(1 - \frac{(x-1)}{n}\right) = 1 \quad \text{and} \quad n\,p = \lambda$$

$$\lim_{n \to \infty} (1-p)^x = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^x = 1$$

$$\lim_{n \to \infty} (1-p)^n = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Applying the above limits , we get,

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty} \frac{n^x \left(1-\frac{1}{n}\right)...\left(1-\frac{(x-1)}{n}\right)}{x!}\, p^x\,(1-p)^{n-x}$$

$$f(x) = \frac{\lambda^x\, e^{-\lambda}}{x!}\,,\ x = 0,1,2,...; \lambda > 0;$$

Moments of Poisson Distribution

Mean: $E(X) = \sum x\, P(X=x) = \sum_{x=0}^{\infty} x \quad \frac{\lambda^x\, e^{-\lambda}}{x!}$

$$\Rightarrow \lambda\, e^{-\lambda} \sum_{x=1}^{\infty} x\ \frac{\lambda^{x-1}}{x\,(x-1)!} = \lambda\, e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda\, e^{-\lambda}\, e^{\lambda} = \lambda$$

So, $E(X) = \mu_1' = \lambda$

$$E(X^2) = \sum x^2\, P(X=x) = \sum_{x=0}^{\infty} x^2\ \frac{\lambda^x\, e^{-\lambda}}{x!}$$

$$\Rightarrow \sum_{x=0}^{\infty} (x(x-1)+x)\ \frac{\lambda^x\, e^{-\lambda}}{x!}$$

$$\Rightarrow \lambda^2\, e^{-\lambda} \sum_{x=1}^{\infty} x(x-1)\ \frac{\lambda^{x-2}}{x\,(x-1)\,(x-2)!} + \sum_{x=0}^{\infty} x \quad \frac{\lambda^x\, e^{-\lambda}}{x!}$$

$$\Rightarrow \lambda^2\, e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda = \lambda^2\, e^{-\lambda}\, e^{\lambda} + \lambda = \lambda^2\ + \lambda$$

So, $E(X^2) = \mu_2' = \lambda^2\ + \lambda$

Variance $\mu_2 = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Similarly, $E(X^3) = \mu_3' = \sum x^3\, P(X=x)$

$$\Rightarrow \sum \Big(x\,(x-1)\,(x-2) + 3\,x^2 - 2\,x\Big)\,P(X=x)$$

$$\Rightarrow \lambda^3 + 3\,\lambda^2 + \lambda$$

Similarly, $E(X^4) = \mu_4' = \sum x^4\,P(X=x)$

$$\Rightarrow \lambda^4 + 6\,\lambda^3 + 7\,\lambda^2 + \lambda$$

Moment Generating Function

The m.g.f, $M_X(t) = E(e^{t\,X}) = \sum e^{t\,x}\,P(X=x) = \sum_{x=0}^{\infty} e^{t\,x}\,\frac{\lambda^x\,e^{-\lambda}}{x!}$

$$\Rightarrow e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda\,e^t)^x}{x!} = e^{-\lambda}\,e^{\lambda\,e^t} = e^{\lambda(e^t-1)}$$

$$E(X) = \mu_1' = \left.\frac{d\,M_X(t)}{dt}\right|_{t=0} = \left.\lambda\,e^t\,e^{\lambda(e^t-1)}\right|_{t=0} = \lambda$$

$$E(X^2) = \mu_2' = \left.\frac{d^2\,M_X(t)}{dt^2}\right|_{t=0} = \left.\frac{d}{dt}\left\{\frac{d\,M_X(t)}{dt}\right\}\right|_{t=0}$$

$$\Rightarrow \left.\frac{d}{dt}\left\{\lambda\,e^t\,e^{\lambda(e^t-1)}\right\}\right|_{t=0} = \left.\lambda\,e^t\,e^{\lambda(e^t-1)} + (\lambda\,e^t)^2\,e^{\lambda(e^t-1)}\right|_{t=0}$$

$$\Rightarrow \lambda^2 + \lambda$$

Additive property of the Poisson distribution

Let X and Y be two independent Poisson random variables with parameters λ_1 and λ_2 respectively. Then their sum $X+Y$ also follows Poisson distribution parameter $\lambda_1+\lambda_2$.

Proof: Since $X \sim P(\lambda_1)$, $M_X(t) = e^{\lambda_1(e^t-1)}$

$Y \sim P(\lambda_2)$, $M_Y(t) = e^{\lambda_2(e^t-1)}$

$M_{(X+Y)}(t) = M_X(t)\,M_Y(t)$, Since X and Y are independent.

$\Rightarrow M_{(X+Y)}(t) = e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$

$\Rightarrow$ M.G.F. of $P(\lambda_1+\lambda_2)$, So, $X+Y \sim P(\lambda_1+\lambda_2)$

Ex.: Let X and Y be two independent Poisson random variables such that $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$. Find the variance of $X - 2Y$.

Sol: Let $X \sim P(\lambda_1)$, and $Y \sim P(\lambda_2)$.

Given, $P(X=1) = P(X=2)$ i.e. $\dfrac{\lambda_1^1\, e^{-\lambda_1}}{1!} = \dfrac{\lambda_1^2\, e^{-\lambda_1}}{2!} \Rightarrow \lambda_1 = 2.$

and $n\,\frac{1}{2} = 12 \Rightarrow n = 24$

Given, $P(Y=2) = P(Y=3)$ i.e. $\dfrac{\lambda_2^2\, e^{-\lambda_2}}{2!} = \dfrac{\lambda_2^3\, e^{-\lambda_2}}{3!} \Rightarrow \lambda_1 = 3.$

Therefore, $V(X) = \lambda_1 = 2$, and $V(Y) = \lambda_2 = 3$.

Then, $V(X-2Y) = V(X) + 4\,V(Y)$, Since X and Y are independent.

$V(X-2Y) = 2 + 4 \times 3 = 14.$

Uniform Distributions

Uniform Distribution: The probability function of the uniform distribution with parameter α and β is given by:

$f(x) = \frac{1}{\beta-\alpha}, \quad \text{where } \alpha < x < \beta, \quad \beta > \alpha.$

The Cumulative distribution function $F(x)$ is obtain by:

$$F(X) = \begin{cases} 0 & x \le a; \\ \dfrac{x-\alpha}{\beta-\alpha} & a < x < b; \\ 1 & x \ge b; \end{cases}$$

M.G.F. of Uniform Distribution: Suppose $X \sim U(\alpha, \beta)$, the M.G.F. is given by

$$M_X(t) = \int_\alpha^\beta e^{t\,x}\, f(x)dx = \frac{1}{\beta-\alpha}\int_\alpha^\beta e^{t\,x}dx = \frac{e^{\beta\,t} - e^{\alpha\,t}}{t\,(\beta-\alpha)}, \quad t \ne 0.$$

Characterstic Function of Uniform Distribution: Suppose $X \sim U(\alpha, \beta)$, the Characterstic Function is given by

$$\phi_X(t) = \int_\alpha^\beta e^{itx} f(x)dx = \frac{1}{\beta-\alpha}\int_\alpha^\beta e^{itx}dx = \frac{e^{i\beta t}-e^{i\alpha t}}{it(\beta-\alpha)}, \quad t \neq 0.$$

Moments of Uniform Distribution: Suppose $X \sim U(\alpha, \beta)$, the r^{th} raw moment is given by

$$\mu_r' = \int_\alpha^\beta x^r f(x)dx = \frac{1}{\beta-\alpha}\int_\alpha^\beta x^r dx = \frac{1}{\beta-\alpha}\left(\frac{\beta^{r+1}-\alpha^{r+1}}{r+1}\right)$$

Let, first raw moment is given by Mean= $\mu_1' = \left(\frac{\alpha+\beta}{2}\right)$

Let, Second raw moment is given by $\mu_2' = \left(\frac{\alpha^2+\beta^2+\alpha\beta}{3}\right)$

Variance is obtained by $= \mu_2 = \mu_2' - \mu_1'^2 = \left(\frac{\alpha^2+\beta^2+\alpha\beta}{3}\right) - \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{1}{12}(\beta-\alpha)^2$

Normal Distribution

Normal Distribution: The probability function of the normal distribution with mean μ and standard deviation σ is given by:

$$f(x) = \tfrac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \text{where} -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0.$$

Standard Normal Distribution: Let $Z = (\frac{x-\mu}{\sigma})$ then Z is standard normal distribution with mean 0 and standard deviation 1.

$$f(z) = \tfrac{1}{\sqrt{2\pi}}\, e^{-z^2/2}, \quad \text{where} -\infty < z < \infty.$$

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Properties of normal distribution:

- It is bell shaped and symmetric curve.
- All measures of central tendency are all equal. i.e. mean=median=mode.
- The curve touches the horizontal line only at the infinity.
- On the basis of mean, μ and SD, σ the area of normal curve is distributed as:

  1. $\mu \mp 1\sigma = 68.27\%$
  2. $\mu \mp 2\sigma = 95.45\%$
  3. $\mu \mp 3\sigma = 99.73\%$

- The value on a normal curve lie between two limits i.e. between μ and σ.
- $\mu \mp 1\,\sigma$ covers 68.27% area of the curve. It indicates that in a normal distribution 68.27% of the observations lie within.
- $\mu \mp 2\,\sigma$ covers 95.45% area of the curve. Here μ to $\mu + 2\,\sigma$ covers 47.72% area on the right and μ to $\mu - 2\,\sigma$ covers 47.72% area on the left of mean.
- $\mu \mp 3\,\sigma$ covers 99.73% area of the curve i.e. it covers the entire area of the curve except 0.27% area, which lies outside the curve. It means area on the right of the mean (μ to $\mu + 3\,\sigma$) covers 49.87% and 49.87% area on the left of mean between μ and $\mu - 3\,\sigma$.

Mean and Variance

Let $X \sim N(\mu,\, \sigma)$, then

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)dx = \int_{-\infty}^{\infty} x\, \frac{1}{\sigma\sqrt{2\,\pi}}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}\, dx$$

$$\text{Assume, } z = \frac{(x-\mu)}{\sigma} \Rightarrow x = \mu + z\,\sigma \text{ and } dx = \sigma\, dz$$

$$\Rightarrow \int_{-\infty}^{\infty} (\mu + z\,\sigma)\, \frac{1}{\sigma\sqrt{2\,\pi}}\, e^{-\frac{z^2}{2}}\, \sigma\, dz = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} (\mu + z\,\sigma)\, e^{-\frac{z^2}{2}}\, dz$$

$$\Rightarrow \mu\, \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz + \sigma\, \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}}\, dz = \mu \times 1 + 0 = \mu$$

Since, $z\, e^{-\frac{z^2}{2}}$ is an odd function of z, $\int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}}\, dz = 0$.

Also $\frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz = 1$

Let $X \sim N(\mu,\, \sigma)$, then

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f_X(x)dx = \int_{-\infty}^{\infty} x^2\, \frac{1}{\sigma\sqrt{2\,\pi}}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}\, dx$$

$$\text{Assume, } z = \frac{(x-\mu)}{\sigma} \Rightarrow x = \mu + z\,\sigma \text{ and } dx = \sigma\, dz$$

$$\Rightarrow \int_{-\infty}^{\infty} (\mu + z\,\sigma)^2 \,\frac{1}{\sigma\,\sqrt{2\,\pi}}\, e^{-\frac{z^2}{2}}\, \sigma\, dz = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} (\mu + z\,\sigma)^2\, e^{-\frac{z^2}{2}}\, dz$$

$$\Rightarrow \frac{\mu^2}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz + \frac{\sigma^2}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} z^2\, e^{-\frac{z^2}{2}}\, dz + \frac{2\,\sigma}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}}\, dz$$

Since, $z\, e^{-\frac{z^2}{2}}$ is an odd function of z, $\int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}}\, dz = 0$.

Also $\frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz = 1$

$$\Rightarrow \mu^2 \times 1 + \frac{\sigma^2}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} z^2\, e^{-\frac{z^2}{2}}\, dz + 0$$

$$\Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{2\,\pi}} \int_{0}^{\infty} z^2\, e^{-\frac{z^2}{2}}\, dz \Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{2\,\pi}} \int_{0}^{\infty} z^2\, e^{-\frac{z^2}{2}}\, dz$$

Let $t = \frac{z^2}{2}$, $\Rightarrow dt = z\, dz$, $\Rightarrow dz = \frac{dt}{\sqrt{2\,t}}$

$$\Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{2\,\pi}} \int_{0}^{\infty} 2\,t\, e^{-t}\, \frac{dt}{\sqrt{2\,t}} \Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{1}{2}}\, e^{-t}\, dt$$

$$\Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{3}{2}-1}\, e^{-t}\, dt \Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{\pi}}\, \Gamma\frac{3}{2} \Rightarrow \mu^2 + \frac{2\,\sigma^2}{\sqrt{\pi}}\, \frac{1}{2}\, \Gamma\frac{1}{2}$$

$$\Rightarrow \mu^2 + \frac{\sigma^2}{\sqrt{\pi}}\, \sqrt{\pi} \Rightarrow \mu^2 + \sigma^2$$

$$E(X^2) = \mu^2 + \sigma^2$$

Variance $= E(X^2) - (E(X))^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Moment Generating Function

$$M_X(t) = E(e^{t\,X}) = \int_{-\infty}^{\infty} e^{t\,x}\, f(x)\, dx = \int_{-\infty}^{\infty} e^{t\,x}\, \frac{1}{\sigma\,\sqrt{2\,\pi}}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}\, dx$$

Assume, $z = \frac{(x-\mu)}{\sigma} \Rightarrow x = \mu + z\,\sigma$ and $dx = \sigma\, dz$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(\mu+z\sigma)}\, e^{-\frac{z^2}{2}}\,\sigma\, dz \Rightarrow \frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}+zt\sigma}\, dz$$

$$\Rightarrow \frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2zt\sigma+t^2\sigma^2)+\frac{t^2\sigma^2}{2}}\, dz \Rightarrow \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\, dz$$

$$\Rightarrow \frac{2\, e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\, dz;$$

Put, $u = \frac{(z-t\sigma)^2}{2} \quad \Rightarrow dz = \frac{1}{\sqrt{2u}}\, du$

$$\Rightarrow \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}}\int_{0}^{\infty} u^{-\frac{1}{2}}\, e^{-u}\, du \Rightarrow \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}}\int_{0}^{\infty} u^{\frac{1}{2}-1}\, e^{-u}\, du$$

$$\Rightarrow \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}}\,\Gamma\frac{1}{2} \quad \Rightarrow \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}}\sqrt{\pi} \quad \Rightarrow e^{\mu t+\frac{\sigma^2 t^2}{2}}$$

So, $M_X(t) = e^{\mu t+\frac{\sigma^2 t^2}{2}}$

$$E(X) = \mu_1' = \left.\frac{d\, M_X(t)}{dt}\right|_{t=0} = \frac{d}{dt}\, e^{\mu t+\frac{\sigma^2 t^2}{2}}\Bigg|_{t=0} \Rightarrow (\mu +\, t\,\sigma^2)\, e^{\mu t+\frac{\sigma^2 t^2}{2}}\Bigg|_{t=0}$$

$$E(X) = \mu_1' = \mu$$

$$E(X^2) = \mu_2' = \left.\frac{d^2\, M_X(t)}{dt^2}\right|_{t=0} = \frac{d}{dt}\left\{\frac{d\, M_X(t)}{dt}\right\}\Bigg|_{t=0}$$

$$\Rightarrow \frac{d}{dt}\left\{(\mu +\, t\,\sigma^2)\, e^{\mu t+\frac{\sigma^2 t^2}{2}}\right\}\Bigg|_{t=0}$$

$$\Rightarrow \left\{\sigma^2\, e^{\mu t+\frac{\sigma^2 t^2}{2}} + (\mu +\, t\,\sigma^2)^2\, e^{\mu t+\frac{\sigma^2 t^2}{2}}\right\}\Bigg|_{t=0} = \sigma^2 + \mu^2$$

$$E(X^2) = \mu_2' = \sigma^2 + \mu^2$$

Additive property of the Normal distribution

If X is a $N(\mu_1,\, \sigma_1^2)$ and Y is a $N(\mu_2,\, \sigma_2^2)$ and they are independent then their sum $X+Y$ also follows $N(\mu_1+\mu_2,\, \sigma_1^2+\sigma_2^2)$.

Proof: Since $X \sim N(\mu_1,\, \sigma_1^2)$, $M_X(t) = e^{\mu_1\, t + \frac{\sigma_1^2\, t^2}{2}}$

$Y \sim N(\mu_2,\, \sigma_2^2)$, $M_Y(t) = e^{\mu_2\, t + \frac{\sigma_2^2\, t^2}{2}}$

$M_{(X+Y)}(t) = M_X(t)\, M_Y(t)$, Since X and Y are independent.

$$\Rightarrow M_{(X+Y)}(t) = e^{\mu_1\, t + \frac{\sigma_1^2\, t^2}{2}}\, e^{\mu_2\, t + \frac{\sigma_2^2\, t^2}{2}} = e^{(\mu_1+\mu_2)\, t + \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)}$$

$\Rightarrow$ M.G.F. of $N(\mu_1+\mu_2,\, \sigma_1^2+\sigma_2^2)$, So, $X+Y \sim N(\mu_1+\mu_2,\, \sigma_1^2+\sigma_2^2)$

Note. If $X_1, X_2, ..., X_n$ are n independent normal variates with mean μ_i and Variance $\sigma_i^2;\ i = 1, 2, ..., n$ respectively. Then the variate $Y = \sum_{i=1}^{n} X_i$ is normally distributed with mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

References

- S.C. Gupta, V.K. Kapoor, Fundamentals of Mathematical Statistics, Sultan Chand & Sons.
- A. Kumar, A. Chaudhary, Probability Distribution & Theory of Attributes, Krishna's Educational Publishers.
- A. Kumar, A. Chaudhary, Probability Distribution & Numerical Analysis, Krishna's Educational Publishers.
- K.S. Negi, Biostatistics, Aitbs Publishers.
- V.B. Rastogi, Fundamentals of Biostatistics, ANE Books.